Interesting Integral - Solution

Find a recursive formula for $\int x^n e^x \, dx$. Note that without a recursive formula, this integral would require many integration by parts in a row.

Denote the integral $\int x^n e^x \, dx$ by I_n. Note that the integral requires integration by parts and that $u = x^n$ and $dv = e^x dx$ is a good start. Then $du = nx^{n-1} dx$ and $v = e^x$, so that

$$I_n = \int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx = x^n e^x - nI_{n-1}.$$

This produces the recursive formula

$$I_n = x^n e^x - nI_{n-1}.$$

Note that it reduces any I_n to I_0 eventually. Since

$$I_0 = \int e^x \, dx = e^x,$$

the formula enables one to find the antiderivative.

Using the Formula for I_5

$$\begin{aligned}
I_5 &= x^5 e^x - 5I_4 \\
&= x^5 e^x - 5\left(x^4 e^x - 4I_3\right) \\
&= x^5 e^x - 5x^4 e^x + 20\left(x^3 e^x - 3I_2\right) \\
&= x^5 e^x - 5x^4 e^x + 20x^3 e^x - 60\left(x^2 e^x - 2I_1\right) \\
&= x^5 e^x - 5x^4 e^x + 20x^3 e^x - 60x^2 e^x + 120\left(xe^x - I_0\right) \\
&= x^5 e^x - 5x^4 e^x + 20x^3 e^x - 60x^2 e^x + 120xe^x - 120e^x + C.
\end{aligned}$$